Țițeica's papers on quantum electrodynamics

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Șerban Țițeica (1908–1985) published four papers on quantum electrodynamics, in the early '40s of the 20th century, which will be discussed in the context of modern quantum theory of fields by Professor Alexandru-Sorin Mărculescu, in a study to appear in the *Romanian Reports in Physics,* vol. 73, no. 2 (2021). These four papers, written in French, were issued in journals with a very limited circulation. The goal of the present preprint is to provide an easily accessible, English version of these papers.

Namely, we are speaking about:

1. Contributions à la théorie des positrons [Contributions to the theory of positrons], submitted in September 1940, issued in *Bull. Soc. Roum. Phys*. vol. 41, no. 76 (1940)

2. Contributions à la théorie des positrons (Deuxième note) [Contributions to the theory of positrons (Second note)], issued in *Bull. Soc. Roum. Phys*. vol. 42, no. 77 (1941)

3. Sur le temps propre en mecanique ondulatoire [On the proper time in wave mechanics], issued in *Bull. Sect. Sci. de l'Acad. Roum*., tome XXV-eme, no. 4 (1942)

4. La polarisation du vide [The vacuum polarization], submitted on September 1942, issued in *Bull. Soc. Roum. Phys*. vol. 43, no. 80 (1942).

These papers, worked out in dramatic years, shared a dramatic fate – they remained unnoticed, and their possible contribution to the development of quantum electrodynamics was lost. Hopefully, the remarkable effort of Prof. Mărculescu allows the access of the modern readers to the unique beauty of this work.

Contributions to the theory of positrons (I)

Şerban Ţiţeica

Abstract

Dirac has shown that the density matrix, which describes the vacuum, is not uniquely determined by its singularities. This may be compared with the result obtained by *Hadamard* for the calculation of the fundamental solution of partial differential equations of the second order. The density matrix can be completely determined by using *Hadamard*'s method of descent.

The relativistic wave theory of electron, proposed in 1928 by *Dirac*, was confronted from the very beginning with a seemingly unavoidable difficulty: the existence, for electron, of negative energy levels. It is well known how *Dirac* himself avoided this difficulty: he admits that an electron distribution with all negative energy levels occupied according to the *Pauli* exclusion principle, and all positive energy levels unoccupied, *is unobservable* and it corresponds to what is usually called *vacuum*. Any modification of this state is observable, and corresponds to the presence of particles. There are two possible modifications, which can occur separately or simultaneously: either there are several positive energy levels occupied by electrons which behave in the familiar way as particles with positive energy and negative charge, or there are several unoccupied negative energy levels. It is easy to realize [1] that these 'holes' behave exactly like particles with positive energy and charge opposite to the electron one.

The theory allows also to predict that, under the action of appropriate electromagnetic fields, an electron occupying a negative level will be able to pass to a positive level, being – so to say – two-fold observable: as an electron on the positive level and as a 'hole' on the previously occupied negative level. We know that the experimental discovery of positron has brilliantly confirmed *Dirac*'s ideas and allowed to identify the positrons with 'holes'.

All previous considerations concern the situation when the electromagnetic field is absent, or when it is weak enough to be treated as a small perturbation. But if we try to generalize the elementary theory just described to arbitrary fields, new difficulties occur. After all, in an arbitrary field the energy levels are completely mixed, and it is impossible to say which ones correspond to positive energies, and which to negative energies. The unobservable distribution of electrons, corresponding to the vacuum, cannot be characterized as simply as in the absence of the electromagnetic field. The essential ideas for the development of the theory in this direction are equally due to *Dirac* [2]. We shall summarize them, as they are absolutely necessary for understanding the content of the present paper. Admitting the correctness of *Hartree* approximation, *Dirac* calculates the matrix density associated with an unobservable distribution in the simple case when the field is absent, and finds that this density has characteristic singularities on the light cone having its top at one of the two spacetime points with respect to which it is calculated. He generalizes this result to an arbitrary field and looks for a solution of the wave equation having singularities of the same sort. He finds out, in this way, that all the singular terms are perfectly determined, but that the regular part of the matrix density must satisfy a certain partial derivatives equation. As it is well known, such an equation has infinitely many solutions and one cannot know a priori which is the appropriate one. The choices made by *Dirac* [2] and *Heisenberg* [3] are not identical. It seems that *Heisenberg*'s choice prevailed, but it still shows a certain degree of arbitrariness.

The aforementioned results can be connected to those obtained by *Hadamard* in his researche on the integration of hyperbolic second order partial derivative equations [4]. In order to solve the *Cauchy* problem, *Hadamard* looks for a solution of the equation, named by him elementary solution, and characterized by certain singularities. He finds out that, when the number of independent variables is *odd*, the elementary solution has only algebraic singularities on the light cone, and moreover it is uniquely determined. When the number of independent variables is *even*, the elementary solution also allows for a logarithmic singularity, and contains an additional, regular term, which has to satisfy a certain partial derivative equation. So, there is a large degree of arbitrariness in choosing this solution. To avoid this inconvenience, *Hadamard* uses the trick of 'descent' , which means to take advantage of the results obtained in the case of an odd number of variables, where these results are fully determined, for solving the problem in the case of a number of dimensions smaller by one, consequently even. Actually, *Hadamard* is using this trick only for the solution of the *Cauchy* problem, but it can be also applied to find out the elementary solution.

By taking into account the close relationship existing between the *Dirac* equation and the wave equation with four independent variables, which is the class of equations studied by *Hadamard*, the results of *Dirac* could have been predicted. The singularities are of the same algebraic-logarithmic nature and the degree of indetermination is the same. We therefore expect that the other results of *Hadamard* can be generalized to the case of *Dirac* equation. Among others, we would expect that, for an odd number of independent variables, the matrix density corresponding to the vacuum will be completely determined by its singularities, and that the method of descent will allow to eliminate the arbitrariness occurring for an even number of variables. In particular, there is a chance to obtain a theory of 'holes' without any arbitrariness, by developing first the theory in a fictitious, five-dimensional world, and by subsequently 'descending' to the real, four-dimensional spacetime. The goal of the next pages is to show that these expectations are correct.

1 First case: Absence of the electromagnetic field

Let us consider the general case of a world $S_{2\nu}$ with 2ν independent variables denoted by $t,\ x_1,\ x_2,\ \ldots, x_{2\nu-1}$; the set of spatial variables will be considered as the components of the space vector, $\vec{x}$. Similarly, the vector operator with components

$$\frac{\partial}{\partial x_1},\ \frac{\partial}{\partial x_2},\ \ldots, \frac{\partial}{\partial x_{2\nu-1}}$$

will be represented by the symbol ∇.

Simultaneously we consider the world $S_{2\nu+1}$ with $2\nu+1$ dimensions that is obtained by adding to $S_{2\nu}$ a new space coordinate z.

The Hamiltonian function for a *Dirac* electron in $S_{2\nu}$ takes therefore the form

$$H_0 = c\vec{\alpha}\vec{p} + \beta mc^2 = c\frac{\hbar}{i}\vec{\alpha}\nabla + \beta mc^2 \tag{1}$$

where $\vec{p}$ is the electron momentum, c the speed of light, $\vec{\alpha}$ and β are matrices satisfying the well known commutation relations [5] and $\hbar$ the *Planck constant divided by* 2π. For applying the method of descent it is also necessary to consider the operator

$$M_0 = c\left(\vec{\alpha}\vec{p} + \beta q\right) = c\frac{\hbar}{i}\left(\vec{\alpha}\nabla + \beta\frac{\partial}{\partial z}\right). \tag{2}$$

The *Dirac* equation for the electron is

$$\mathcal{H}_0\psi = 0 \tag{3}$$

where $\mathcal{H}_0$ is the operator

$$\mathcal{H}_0 = \frac{i}{c}\frac{\partial}{\partial t} - \frac{1}{\hbar c}H_0 = i\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\alpha}\nabla\right) - \beta\frac{mc}{\hbar}. \tag{4}$$

At the same time we consider the equation

$$\mathcal{M}_0\varkappa = 0 \tag{5}$$

with

$$\mathcal{M}_0 = \frac{i}{c}\frac{\partial}{\partial t} - \frac{1}{\hbar c}M_0 = i\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\alpha}\nabla + \beta\frac{\partial}{\partial z}\right). \tag{6}$$

The equations (3) and (4) give

$$\psi\left(t, \vec{x}\right) = e^{-\frac{i}{\hbar}H_0 t}\,\psi\left(0, \vec{x}\right) \tag{7}$$

and

$$\varkappa\left(t, \vec{x}, z\right) = e^{-\frac{i}{\hbar}M_0 t}\,\varkappa\left(0, \vec{x}, z\right), \tag{8}$$

respectively.

Let us take for $\psi(0,\vec{x})$ and $\varkappa(t,\vec{x},z)$ the plane waves

$$\psi\left(0,\vec{x};\vec{k}\right) = \frac{1}{(2\pi)^{\nu-\frac{1}{2}}} e^{i\vec{k}\vec{x}};$$

$$\varkappa\left(0,\vec{x},z;\vec{k},l\right) = \frac{1}{(2\pi)^{\nu}} e^{i\left(\vec{k}\vec{x}+lz\right)}$$

normalized such that

$$\int \psi^*\left(0,\vec{x};\vec{k'}\right)\psi\left(0,\vec{x};\vec{k''}\right)\mathrm{d}\vec{x} = \delta\left(\vec{k'}-\vec{k''}\right)$$

$$\int \varkappa^*\left(0,\vec{x},z;\vec{k'},l'\right)\varkappa\left(0,\vec{x},z;\vec{k''},l''\right)\mathrm{d}\vec{x}\mathrm{d}z = \delta\left(\vec{k'}-\vec{k''}\right)\delta\left(l'-l''\right).$$

In this case, the wave functions $\psi\left(0,\vec{x};\vec{k}\right)$ and $\varkappa\left(0,\vec{x},z;\vec{k},l\right)$ will be further given by the expressions (7) and (8) if one agrees to replace the operations ∇ and $\dfrac{\partial}{\partial z}$ by $i\vec{k}$ and il, respectively, that is

$$\psi\left(t,\vec{x};\vec{k}\right) = e^{-ic\left(\vec{\alpha}\vec{k}+\beta\frac{mc}{\hbar}\right)t}\frac{1}{(2\pi)^{\nu-\frac{1}{2}}}e^{i\vec{k}\vec{x}},$$

$$\varkappa\left(t,\vec{x},z;\vec{k},l\right) = e^{-ic\left(\vec{\alpha}\vec{k}+\beta l\right)t}\frac{1}{(2\pi)^{\nu}}e^{i\left(\vec{k}\vec{x}+lz\right)}.$$

Let us observe now that

$$\left(\vec{\alpha}\vec{k}+\beta\frac{mc}{\hbar}\right)^2 = k^2+\left(\frac{mc}{\hbar}\right)^2 \tag{9}$$

and

$$\left(\vec{\alpha}\vec{k}+\beta l\right)^2 = k^2+l^2. \tag{10}$$

By expanding in power series the exponential having operators as exponent one immediately obtains

$$e^{-ict\left(\vec{\alpha}\vec{k}+\beta\frac{mc}{\hbar}\right)} = \cos\left(ct\sqrt{k^2+\mu^2}\right) - i\sin\left(ct\sqrt{k^2+\mu^2}\right)\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}$$

$$= \frac{1}{2}\left(1-\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}\right)e^{ict\sqrt{k^2+\mu^2}} + \frac{1}{2}\left(1+\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}\right)e^{-ict\sqrt{k^2+\mu^2}}$$

where $\mu = \dfrac{mc}{\hbar}$ is the *Compton* wave number. The other exponential is similarly expanded, and the result can be obtained replacing μ by the variable l.

If one agrees to always take the positive value of the square root one finds that the wave function splits into two parts, a part associated with positive energies and another one with negative energies. These two parts are labelled by the lower subscripts + and − as indicated below:

$$\psi_+\left(t,\vec{x};\vec{k}\right)=\frac{1}{2\left(2\pi\right)^{\nu-\frac{1}{2}}}\left(1+\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}\right)e^{i\left(\vec{k}\vec{x}-ct\sqrt{k^2+\mu^2}\right)}$$

$$\psi_-\left(t,\vec{x};\vec{k}\right)=\frac{1}{2\left(2\pi\right)^{\nu-\frac{1}{2}}}\left(1-\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}\right)e^{i\left(\vec{k}\vec{x}+ct\sqrt{k^2+\mu^2}\right)}$$

$$\varkappa_\pm\left(t,\vec{x},z;\vec{k},l\right)=\frac{1}{2\left(2\pi\right)^{\nu}}\left(1\pm\frac{\vec{\alpha}\vec{k}+\beta l}{\sqrt{k^2+l^2}}\right)e^{i\left(\vec{k}\vec{x}+lz\mp ct\sqrt{k^2+l^2}\right)}.$$

From these expressions and the equations (9) and (10) one can immediately obtain the *Dirac* matrix densities for the two worlds with 2ν and $2\nu+1$ dimensions

$$\Sigma_{2\nu}^{-}=\int\psi_-\left(t',\vec{x'};\vec{k}\right)\psi_-^*\left(t'',\vec{x''};\vec{k}\right)\mathrm{d}\vec{k}=$$

$$=\frac{1}{2}\frac{1}{\left(2\pi\right)^{2\nu-1}}\int\left(1-\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}\right)e^{i\left(\vec{k}\vec{x}+ct\sqrt{k^2+\mu^2}\right)}\mathrm{d}\vec{k}$$

$$\Sigma_{2\nu}^{+}=\int\psi_+\left(t',\vec{x'};\vec{k}\right)\psi_+^*\left(t'',\vec{x''};\vec{k}\right)\mathrm{d}\vec{k}=$$

$$=\frac{1}{2}\frac{1}{\left(2\pi\right)^{2\nu-1}}\int\left(1+\frac{\vec{\alpha}\vec{k}+\beta\mu}{\sqrt{k^2+\mu^2}}\right)e^{i\left(\vec{k}\vec{x}-ct\sqrt{k^2+\mu^2}\right)}\mathrm{d}\vec{k}$$

$$\Sigma_{2\nu+1}^{\mp}=\int\varkappa_\mp\left(t',\vec{x'},z';\vec{k},l\right)\varkappa_\mp^*\left(t'',\vec{x''},z'';\vec{k},l\right)\mathrm{d}\vec{k}\mathrm{d}l=$$

$$=\frac{1}{2}\frac{1}{\left(2\pi\right)^{2\nu}}\int\left(1\mp\frac{\vec{\alpha}\vec{k}+\beta l}{\sqrt{k^2+l^2}}\right)e^{i\left(\vec{k}\vec{x}+lz\pm ct\sqrt{k^2+l^2}\right)}\mathrm{d}\vec{k}\mathrm{d}l.$$

In these expressions it has been put $\vec{x'}-\vec{x''}=\vec{x},\ z'-z''=z$, and $t'-t''=t$.

By a simple transformation these integrals can be brought to the following form:

$$\Sigma_{2\nu}^{\mp}=\pm\mathcal{G}_0\frac{1}{2}\frac{1}{\left(2\pi\right)^{2\nu-1}}\int\frac{e^{i\left(\vec{k}\vec{x}\pm ct\sqrt{k^2+\mu^2}\right)}}{\sqrt{k^2+\mu^2}}\mathrm{d}\vec{k}$$

$$\Sigma_{2\nu+1}^{\mp}=\pm\mathcal{N}_0\frac{1}{2}\frac{1}{\left(2\pi\right)^{2\nu}}\int\frac{e^{i\left(\vec{k}\vec{x}+lz\pm ct\sqrt{k^2+l^2}\right)}}{\sqrt{k^2+l^2}}\mathrm{d}\vec{k}\mathrm{d}l$$

where $\mathcal{G}_0$ and $\mathcal{N}_0$ denote the operators

$$\mathcal{G}_0 = \frac{1}{i}\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\alpha}\nabla\right) - \beta\mu$$

and

$$\mathcal{N}_0 = \frac{1}{i}\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\alpha}\nabla - \beta\frac{\partial}{\partial z}\right),$$

respectively.

The densities of interest in positron theory read:

$$\Sigma_{2\nu}^{-} - \Sigma_{2\nu}^{+} = \mathcal{G}_0 \frac{1}{(2\pi)^{2\nu-1}} \int e^{i\vec{k}\vec{x}} \frac{\cos\left(ct\sqrt{k^2+\mu^2}\right)}{\sqrt{k^2+\mu^2}} \mathrm{d}\vec{k} \tag{11}$$

and

$$\Sigma_{2\nu+1}^{-} - \Sigma_{2\nu+1}^{+} = \mathcal{N}_0 \frac{1}{(2\pi)^{2\nu}} \int e^{i\left(\vec{k}\vec{x}+lz\right)} \frac{\cos\left(ct\sqrt{k^2+l^2}\right)}{\sqrt{k^2+l^2}} \mathrm{d}\vec{k}\mathrm{d}l. \tag{12}$$

One can immediately find the relation between the two densities, thereby providing a 'descent' recipe in absence of the electromagnetic field: one can pass from the density in $2\nu+1$ dimensions to the density in 2ν dimensions by replacing in $\mathcal{N}_0$ the operator $\frac{1}{i}\frac{\partial}{\partial z}$ with μ, multiplying then the quantity under $\mathcal{N}_0$ by $e^{-i\mu z}$ and integrating over z from $-\infty$ to $+\infty$. Indeed, this integration removes the factor e^{ilz} and replaces it by $2\pi\delta\left(l-\mu\right)$. The factor 2π lowers with one unit the value of its exponent in the denominator, and the integration with respect to l replaces everywhere l with μ.

Let us work out this calculation, which will provide us useful insights into the case when a certain field is present. For this purpose we introduce in the spaces $\vec{x},\ z$ and $\vec{k},\ l$ the polar coordinates with radii

$$R = \sqrt{x_1^2 + x_2^2 + \cdots + x_{2\nu-1}^2 + z^2} = \sqrt{r^2+z^2}$$

and

$$K = \sqrt{k_1^2 + k_2^2 + \cdots + k_{2\nu-1}^2 + l^2} = \sqrt{k^2+l^2}.$$

One gets this way

$$\begin{aligned} I' &= \frac{1}{(2\pi)^{2\nu}} \int e^{i\left(\vec{k}\vec{x}+lz\right)} \frac{\cos\left(ct\sqrt{k^2+l^2}\right)}{\sqrt{k^2+l^2}} \mathrm{d}\vec{k}\mathrm{d}l = \\ &= \frac{1}{(2\pi)^{2\nu}} \int e^{iKR\cos\theta} \cos\left(ctK\right) K^{2\nu-2} \mathrm{d}K\mathrm{d}\Omega \end{aligned}$$

where θ is the angle between the vectors $\vec{k},\ l$ and $\vec{x},\ z$, and $d\Omega$ is the infinitesimal solid angle in 2ν dimensions. Integration with respect to Ω gives

$$\frac{1}{(2\pi)^{2\nu}} \int e^{iKR\cos\theta} \mathrm{d}\Omega = \frac{1}{(2\pi)^{\nu}} \frac{J_{\nu-1}(KR)}{(KR)^{\nu-1}}$$

where $J_{\nu-1}$ denotes the *Bessel* function of order $\nu - 1$. The result is

$$I' = \frac{1}{(2\pi)^{\nu}} \int \frac{J_{\nu-1}(KR)}{(KR)^{\nu-1}} \cos(ctK)\, K^{2\nu-1} \mathrm{d}K.$$

By taking into account the recurrence relation for *Bessel* functions

$$\begin{aligned} \frac{J_{\nu-1}(KR)}{(KR)^{\nu-1}} &= -\frac{1}{K^2} \frac{1}{R} \frac{\partial}{\partial R} \left[\frac{J_{\nu-2}(KR)}{(KR)^{\nu-2}} \right] = \cdots \\ &= \frac{1}{K^{2(\nu-1)}} \left(-\frac{1}{R} \frac{\partial}{\partial R} \right)^{\nu-1} J_0(KR), \end{aligned}$$

the integral becomes

$$I' = \frac{1}{(2\pi)^{\nu}} \left(-\frac{1}{R} \frac{\partial}{\partial R} \right)^{\nu-1} \int_0^{\infty} J_0(KR) \cos(ctK)\, \mathrm{d}K.$$

But one has

$$\int_0^{\infty} J_0(KR) \cos(ctK)\, \mathrm{d}K = \begin{cases} \dfrac{1}{\sqrt{R^2 - c^2t^2}} & \text{if } R^2 - c^2t^2 > 0 \\ 0 & \text{if } R^2 - c^2t^2 < 0 \end{cases},$$

hence

$$\begin{aligned} I' &= \frac{1}{(2\pi)^{\nu}} \left(-\frac{1}{R} \frac{\partial}{\partial R} \right)^{\nu-1} \left\{ \begin{array}{c} \dfrac{1}{\sqrt{R^2 - c^2t^2}} \\ 0 \end{array} \right\} \\ &= \frac{1}{(2\pi)^{\nu}} \left(-\frac{1}{r} \frac{\partial}{\partial r} \right)^{\nu-1} \left\{ \begin{array}{c} \dfrac{1}{\sqrt{z^2 + r^2 - c^2t^2}} \\ 0 \end{array} \right\} \end{aligned} \tag{13}$$

with $r = \sqrt{x_1^2 + x_2^2 + \ldots + x_{2\nu-1}^2}$.

Let us now work out the descent

$$\begin{aligned} I &= \frac{1}{(2\pi)^{2\nu-1}} \int e^{i\vec{k}\vec{x}} \frac{\cos\left(ct\sqrt{k^2+\mu^2}\right)}{\sqrt{k^2+\mu^2}} \mathrm{d}\vec{k} \\ &= \frac{1}{(2\pi)^{\nu}} \left(-\frac{1}{r} \frac{\partial}{\partial r} \right)^{\nu-1} \int_{-\infty}^{+\infty} \frac{e^{-i\mu z} \mathrm{d}z}{\sqrt{z^2 - (c^2t^2 - r^2)}} \end{aligned}$$

for $r^2 - c^2t^2 > 0$, and

$$I = \frac{1}{(2\pi)^\nu}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^{\nu-1} \left[\int_{-\infty}^{-\sqrt{c^2t^2-r^2}} \frac{e^{-i\mu z}\mathrm{d}z}{\sqrt{z^2+r^2-c^2t^2}} + \int_{+\sqrt{c^2t^2-r^2}}^{+\infty} \frac{e^{-i\mu z}\mathrm{d}z}{\sqrt{z^2+r^2-c^2t^2}}\right] \qquad (14)$$

for $c^2t^2 - r^2 > 0$.

The theory of cylindric functions then gives

$$I = \frac{1}{(2\pi)^\nu}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^{\nu-1} \begin{cases} \pi i H_0^{(1)}\left(i\mu\sqrt{r^2-c^2t^2}\right) & \text{if } r^2-c^2t^2>0 \\ -\pi N_0\left(\mu\sqrt{c^2t^2-r^2}\right) & \text{if } c^2t^2-r^2>0 \end{cases} \qquad (15)$$

where $H_0^{(1)}$ and N_0 are the well known *Hankel* and *Neumann* functions. This result is up to notations identical to that obtained by *Dirac* [2] for the case $\nu = 2$. If one had reversed the order of derivation and integration operations in (14) one would have been confronted with divergent integrals. One can however follow this approach and get the same result by taking the 'finite part' (see *Hadamard*[4]) of divergent integrals or, equivalently, by conveniently deforming the integration path in the complex plane of the variable z. We restrict ourselves to these short remarks as we shall come back to this method when we shall work out the descent in the presence of an arbitrary field.

2 Second case: the presence of an electromagnetic field

In the world $S_{2\nu}$ the field is defined by a scalar potential U and a vector potential $\vec{A} = A_1,\ A_2, ..., A_{2\nu-1}$. These potentials depend on the variables t and $\vec{x}$. In going to the universe $S_{2\nu+1}$, we assume that the additional component A_z is *identically zero* and that the other potentials *do not depend* on z.

The Hamiltonian function for a *Dirac* electron in $S_{2\nu}$ is

$$H = c\vec{\alpha}\vec{p} + \beta mc^2 - \mathrm{e}\left(U - \vec{\alpha}\vec{A}\right)$$

and the wave equation becomes

$$\mathcal{H}\psi = 0 \qquad (16)$$

with

$$\mathcal{H} = i\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\alpha}\nabla\right) - \beta\mu + \frac{\mathrm{e}}{\hbar c}\left(U - \vec{\alpha}\vec{A}\right). \qquad (17)$$

We shall also consider the following operators:

$$\begin{aligned}
\mathcal{G} &= \frac{1}{i}\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\alpha}\nabla\right) - \beta\mu - \frac{\mathrm{e}}{\hbar c}\left(U + \vec{\alpha}\vec{A}\right), && (18)\\
\mathcal{M} &= \frac{1}{i}\left(\frac{1}{c}\frac{\partial}{\partial t} + \vec{\alpha}\nabla + \beta\frac{\partial}{\partial z}\right) + \frac{\mathrm{e}}{\hbar c}\left(U - \vec{\alpha}\vec{A}\right), && (19)\\
\mathcal{N} &= \frac{1}{i}\left(\frac{1}{c}\frac{\partial}{\partial t} - \vec{\alpha}\nabla - \beta\frac{\partial}{\partial z}\right) - \frac{\mathrm{e}}{\hbar c}\left(U + \vec{\alpha}\vec{A}\right). && (20)
\end{aligned}$$

One requires a solution of equation (16) having the same singularities as the expression (11) of the previous paragraph, that is for a function $\psi\left(t', \vec{x'}; t'', \vec{x''}\right)$ of the variables $t', \vec{x'}$ and of the parameters $t'', \vec{x''}$ satisfying the equation (16) with respect to the variables $t', \vec{x'}$, and having the same singularities as (11)on the light cone with the top at $t'', \vec{x''}$.

The method of descent consists of looking first for a solution $\varkappa$ of the equation

$$\mathcal{M}\varkappa = 0 \tag{21}$$

which is a function of the variables $t',\ \vec{x'},\ z'$ and of the parameters $t'', \vec{x''}, z''$, and has on the light cone with the top at $t'', \vec{x''}, z''$ the same singularities as (11).

The calculations of the previous paragraph prompt us to set

$$\varkappa = \mathcal{N}\omega \tag{22}$$

where the operator $\mathcal{N}$ acts on the variables $t',\ \vec{x'},\ z'$. We also have

$$\mathcal{M}\mathcal{N}\omega = 0. \tag{23}$$

The function ω must have the same singularities as the expression (13)

$$I' = \frac{1}{(2\pi)^\nu}\left(-\frac{1}{R}\frac{\partial}{\partial R}\right)^{\nu-1}\frac{1}{\sqrt{R^2 - c^2t^2}}.$$

We therefore introduce

$$\omega = \frac{f\left(t', \vec{x'}, z'; t'', \vec{x''}, z''\right)}{\left(R^2 - c^2t^2\right)^{\nu-\frac{1}{2}}} \tag{24}$$

with f a regular function on the light cone. Again, as in the previous paragraph we have put $\vec{x} = \vec{x'} - \vec{x''},\ z = z' - z'',\ t - t' - t'', R = \sqrt{x_1^2 + \cdots + x_{2\nu-1}^2 + z^2}$.

Before substituting the expression (24) into equation (23), let us remark that by a calculation almost similar to that usually done for obtaining the second order *Dirac* equation, one finds

$$\mathcal{M}\mathcal{N} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 - \frac{\partial^2}{\partial z^2} - 2\frac{i\mathrm{e}}{\hbar c}\left(U\frac{1}{c}\frac{\partial}{\partial t} + \vec{A}\nabla\right) + \cdots,$$

with $\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \cdots + \frac{\partial^2}{\partial x_{2\nu-1}^2}$, and where the dots indicate terms without derivatives acting on the function f. An easy calculation leads to

$$\mathcal{M}\mathcal{N}\frac{f}{\left(R^2-c^2t^2\right)^{\nu-\frac{1}{2}}} = \frac{1}{\left(R^2-c^2t^2\right)^{\nu+\frac{1}{2}}}\left\{\left(R^2-c^2t^2\right)\mathcal{M}\mathcal{N}f\right.$$

$$\left.+2\left(2\nu-1\right)\left[t\frac{\partial}{\partial t'} + \vec{x}\nabla' + z\frac{\partial}{\partial z'} - \frac{\mathrm{ie}}{\hbar c}\left(Uct - \vec{A}\vec{x}\right)\right]f\right\} = 0. \qquad (25)$$

(The arguments of the functions U and $\vec{A}$ are t' and $\vec{x'}$). We see that our calculation is somehow complementary to the calculation done by *Hadamard*. The difference is that *Hadamard* is looking for the values taken by the singular solution inside the light cone, which has the effect of introducing the root $\sqrt{c^2t^2-R^2}$, while in our problem the calculation worked out in the first section compels us to consider the singular solution only outside the light cone. His approache is however immediately applicable to our problem, and we shall use it in order to find the solution f of the equation (25). For this purpose let us set

$$f = f_0 + \left(R^2-c^2t^2\right)f_1 + \left(R^2-c^2t^2\right)^2 f_2 + \cdots + \left(R^2-c^2t^2\right)^p f_p + \ldots. \qquad (26)$$

Introducing this expression in equation (25) and equating to zero the coefficients of the powers of $R^2 - c^2t^2$ one obtains at first the equation for f_0

$$\left(t\frac{\partial}{\partial t'} + \vec{x}\nabla' + z\frac{\partial}{\partial z'}\right)f_0 = \frac{\mathrm{ie}}{\hbar c}\left(Uct - \vec{A}\vec{x}\right)f_0.$$

One integrates this equation by setting

$$t' = t'' + \tau s,\ \ \vec{x'} = \vec{x''} + \vec{\xi}s,\ \ z' = z'' + \zeta s$$

with s a parameter and $\tau,\ \vec{\xi},\ \zeta$ constants. Since

$$t\frac{\partial}{\partial t'} + \vec{x}\nabla' + z\frac{\partial}{\partial z'} = s\frac{\mathrm{d}}{\mathrm{d}s}$$

holds, one finds for f_0

$$\frac{\mathrm{d}f_0}{\mathrm{d}s} = \frac{\mathrm{ie}}{\hbar c}\left(Uc\tau - \vec{A}\vec{\xi}\right)f_0.$$

This equation can be immediately integrated and gives

$$\begin{aligned} f_0 &= C\exp\left[\frac{ie}{\hbar c}\int_0^s\left(Uc\tau - \vec{A}\vec{\xi}\right)\mathrm{d}\bar{s}\right] \\ &= C\exp\left[\frac{ie}{\hbar c}\int_{t'',\vec{x''}}^{t',\vec{x'}}\left(Uc\mathrm{d}t - \vec{A}\mathrm{d}\vec{x}\right)\right] \end{aligned} \qquad (27)$$

where C is an arbitrary constant, and the integral in the exponent must be taken along the line joining the points $t'', \vec{x''}$ and $t', \vec{x'}$. Notice that in order to avoid any confusion we denote the (negative) electric charge of the electron by $-\mathrm{e}$ and by e the base of the natural logarithm.

The functions f_p of higher order ($p > 0$) are determined from the following recurrence relations:

$$\left(t\frac{\partial}{\partial t'} + \vec{x}\nabla' + z\frac{\partial}{\partial z'}\right) f_p - \frac{i\mathrm{e}}{\hbar c}\left(Uct - \vec{A}\vec{x}\right) f_p + p f_p =$$

$$= \frac{\mathcal{M}\mathcal{N} f_{p-1}}{4\left(p - \nu + \frac{1}{2}\right)}.$$

We observe now that f_0, given by (27), does not depend on z. As z enters the latter recurrence relation only through the operation $\dfrac{\partial}{\partial z'}$ none of the subsequent functions will depend on z. The relation then simplifies and becomes

$$\left(t\frac{\partial}{\partial t'} + \vec{x}\nabla'\right) f_p - \frac{i\mathrm{e}}{\hbar c}\left(Uct - \vec{A}\vec{x}\right) f_p + p f_p = \frac{\overline{\mathcal{M}\mathcal{N}} f_{p-1}}{4\left(p - \nu + \frac{1}{2}\right)}$$

where $\overline{\mathcal{M}\mathcal{N}}$ represents the operator $\mathcal{M}\mathcal{N}$ with the term $\dfrac{\partial^2}{\partial z^2}$ omitted. One more simplification occurs by putting

$$f_p = \frac{\omega_p}{p!\, 2^{2p}\Gamma\left(p - \nu + \frac{3}{2}\right)}.$$

One gets this way

$$\left(t\frac{\partial}{\partial t'} + \vec{x}\nabla'\right)\omega_p - \frac{i\mathrm{e}}{\hbar c}\left(Uct - \vec{A}\vec{x}\right)\omega_p + p\omega_p$$

$$= p\overline{\mathcal{M}\mathcal{N}}\omega_{p-1}. \tag{28}$$

The same substitution made in order to obtain f_0 gives here

$$s\frac{\mathrm{d}\omega_p}{\mathrm{d}s} - \frac{i\mathrm{e}}{\hbar c} s\left(Uc\tau - \vec{A}\vec{\xi}\right)\omega_p + p\omega_p = p\overline{\mathcal{M}\mathcal{N}}\omega_{p-1}.$$

Taking into account that

$$s\frac{\mathrm{d}\omega_0}{\mathrm{d}s} - \frac{i\mathrm{e}}{\hbar c} s\left(Uc\tau - \vec{A}\vec{\xi}\right)\omega_0 = 0$$

one readily finds the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\omega_p}{\omega_0}\right) + \frac{p}{s}\left(\frac{\omega_p}{\omega_0}\right) = \frac{p}{s}\frac{\overline{\mathcal{M}\mathcal{N}}\omega_{p-1}}{\omega_0}$$

whose integral reads:

$$\omega_p = \omega_0 s^{-p}\int_0^s \frac{\overline{\mathcal{M}\mathcal{N}}\omega_{p-1}}{\omega_0}\mathrm{d}\left(\bar{s}^p\right). \tag{29}$$

It is easy to see now that the result depends only on t', $\vec{x'}$, t'' and $\vec{x''}$. Hence

$$\omega = C \sum_{p=0}^{\infty} \frac{\omega_p}{p!\, 2^{2p}\, \Gamma\left(p - \nu + \frac{3}{2}\right)} \left(R^2 - c^2 t^2\right)^{p-\nu+\frac{1}{2}} . \tag{30}$$

Two remarks concernig this result are here in order. First of all, it is obviously valid only for functions U and $\vec{A}$ analytical with respect to their independent variables. *Hadamard* gets rid of this restriction by showing that it is sufficient if these functions admit derivatives of high enough order to allow the calculation of all the terms with negative exponent $p - \nu + \frac{1}{2}$. His results cannot be directly transferred to our problem, on one hand because of the spin variables, implicitely involved in the function ω, and on the other hand because *Hadamard* does not demonstrate his result for the singular solution itself, but for the solution of the *Cauchy* problem. However, we think that the analogy goes far enough to remove, in the case under consideration, the troublesome analyticity condition, and we hope to come back to this issue in a forthcoming publication.

Let us remark next that for $U = 0$ and $\vec{A} = 0$ the series (30) is reduced at its first term. Indeed, ω_0 is just a constant, so $\overline{\mathcal{M}\mathcal{N}}$ becomes the operator $\mathcal{M}_0\mathcal{N}_0$ defined in the previous section and contains only derivative operations giving zero when applied to a constant. Step by step one can show this way that all ω_p with $p > 0$ vanish. If ω_0 is normalized to

$$\omega_0 = \exp\left[\frac{i\mathrm{e}}{\hbar c} \int_{t'',\vec{x''}}^{t',\vec{x'}} \left(U c\mathrm{d}t - \vec{A}\mathrm{d}\vec{x}\right)\right] \tag{31}$$

ω reduces for a zero field to the value

$$\frac{C}{\Gamma\left(\frac{3}{2} - \nu\right)} \frac{1}{\left(R^2 - c^2 t^2\right)^{\nu - \frac{1}{2}}} .$$

In order that this expression equals I' (see (13)) the following condition has to be fulfilled:

$$\frac{C}{\Gamma\left(\frac{3}{2} - \nu\right)} = \frac{1}{2} \frac{1}{\pi^\nu} \frac{\Gamma\left(\nu - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}\right)},$$

that is

$$C = \frac{1}{2} \frac{1}{\pi^\nu} \frac{\Gamma\left(\nu - \frac{1}{2}\right) \Gamma\left(\frac{3}{2} - \nu\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{(-1)^{\nu-1}}{2\pi^{\nu - \frac{1}{2}}},$$

and

$$\omega = \frac{(-1)^{\nu-1}}{2\pi^{\nu-\frac{1}{2}}} \sum_{p=1}^{\infty} \frac{\omega_p}{p!\, 2^{2p}\, \Gamma\left(p - \nu + \frac{3}{2}\right)} \left(R^2 - c^2 t^2\right)^{p-\nu+\frac{1}{2}} . \tag{32}$$

Obviously, for $c^2 t^2 - R^2 > 0$, one has to put $\omega = 0$. The matrix density describing the vacuum in $S_{2\nu+1}$ is then

$$\varkappa = \mathcal{N}\omega,$$

and as expected it is perfectly determined by its singularities.

In order to perform the descent to the $S_{2\nu}$-dimensional world one has just to put

$$\psi = \mathcal{G}\varphi \tag{33}$$

with $\mathcal{G}$ being the operator (18), and φ defined by

$$\varphi = \int_{-\infty}^{\infty} e^{-i\mu z}\omega \mathrm{d}z.$$

Since ω depends on z only through the expression $R^2 - c^2t^2 = r^2 + z^2 - c^2t^2$ we have to compute the following integrals:

$$C_p = \int_{-\infty}^{\infty} \left(z^2 + r^2 - c^2t^2\right)^{p-\nu+\frac{1}{2}} e^{-i\mu z}\omega \mathrm{d}z \tag{34}$$

for $r^2 - c^2t^2 > 0$, and

$$D_p = \left\{ \int_{-\infty}^{-\sqrt{c^2t^2-r^2}} + \int_{\sqrt{c^2t^2-r^2}}^{\infty} \right\} \left[z^2 - \left(c^2t^2 - r^2\right)\right]^{p-\nu+\frac{1}{2}} e^{-i\mu z} \mathrm{d}z \tag{35}$$

for $c^2t^2 - r^2 > 0$.

These integrals are in general divergent, but one can, through a convenient deformation of the integration path in the complex plane of the variable z unambiguously define a 'finite part'.

Let us first examine C_p that is convergent for $p < \nu$. The function to be integrated has two branch points at $z = \pm i\sqrt{r^2 - c^2t^2}$. It becomes one-valued if one cuts the z plane along two half-lines starting at the critical points and ending at infinity (see Fig 1). One can without changing the value of the integral deform the integration path such that it wraps the cut starting at the point $-i\sqrt{r^2 - c^2t^2}$, provided the branches of the integrand reduce to their arithmetical values along the real axis. The integral taken along the new integration path remains convergent for any finite p and defines the 'finite part' of C_p. One obtains then

$$C_p = i\pi \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu - p - \frac{1}{2}\right)} 2^{p-\nu+1} \left(\frac{i\sqrt{r^2 - c^2t^2}}{\mu}\right)^{p-\nu+1} H^{(1)}_{p-\nu+1}\left(i\mu\sqrt{r^2 - c^2t^2}\right)$$

where $H^{(1)}$ is the first Hankel function and i has the value $i = e^{i\frac{\pi}{2}}$.

Let us consider now the integrals D_p which have $z = \pm\sqrt{c^2t^2 - r^2}$ as critical points. They are convergent only for $p = \nu - 1$, at least if one is limiting oneself to integer values of p. In this case one can without changing the value of the integrals deform the integration path into two lace contours bypassing the critical points clockwise, continuing parallely to the negative imaginary z-axis, and taking afterwards one half of the result (see Fig. 2). The function to

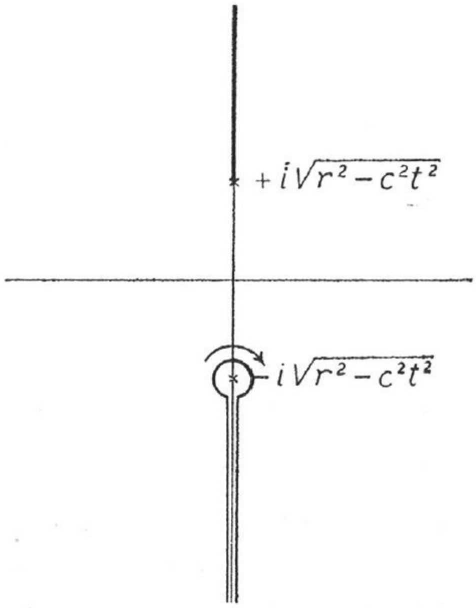

Figure 1:

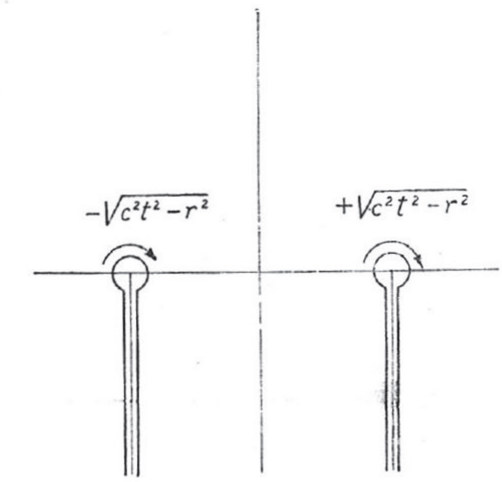

Figure 2:

be integrated has to take values reducing to the arithmetical values along the half-axes $z > +\sqrt{c^2t^2 - r^2}$ and $z < -\sqrt{c^2t^2 - r^2}$. One can prove this by first taking as integration path the laces parallel to the real axis, doubling therefore the value of the integral, hence the need of taking a half of the result. After that, one turns the laces around the critical points until the suitable positions are reached. The integrals D_p are then convergent for any p and their values define the 'finite parts'. Finally, one finds

$$D_p = -\pi \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu - p - \frac{1}{2}\right)} 2^{p-\nu+1} \left(\frac{\sqrt{c^2t^2 - r^2}}{\mu}\right)^{p-\nu+1} N_{p-\nu+1}\left(\mu\sqrt{c^2t^2 - r^2}\right)$$

where N is the *Neumann* function.

The theory of *Bessel* functions gives

$$N_n(u) = \frac{2}{\pi} J_n(u) \ln u + f(u)$$

with f an one-valued function, and n an integer. By putting $u = iv$ with v positive, and $i = e^{i\frac{\pi}{2}}$, one gets

$$N_n(iv) = \frac{2}{\pi} J_n(iv) \ln v + f(iv) + iJ_n(iv),$$

hence

$$\begin{aligned} H_n^{(1)}(iv) &= J_n(iv) + iN_n(iv) \\ &= i\left[\frac{2}{\pi} J_n(iv) \ln v + f(iv)\right] \end{aligned}$$

such that

$$-iH_n^{(1)}(iv) = \frac{2}{\pi} J_n(iv) \ln|iv| + f(iv).$$

The integral D_p becomes thus identical to C_p if one agrees to replace in the logarithmic term the root $\sqrt{c^2t^2 - r^2}$ by the root $\sqrt{|c^2t^2 - r^2|}$. Having accepted this convention the expression of φ becomes

$$\begin{aligned} \varphi &= \frac{(-1)^{\nu-1}}{2\pi^{\nu-\frac{1}{2}}} \sum_{p=0}^{\infty} \frac{\omega_p}{p!\, 2^{2p}\, \Gamma\left(p - \nu + \frac{3}{2}\right)} \left[-\pi \frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\nu - p - \frac{1}{2}\right)}\right] \\ &\quad \cdot \left(2\frac{\sqrt{c^2t^2 - r^2}}{\mu}\right)^{p-\nu+1} N_{p-\nu+1}\left(\mu\sqrt{c^2t^2 - r^2}\right) \\ &= -\frac{1}{2} \frac{1}{(2\pi)^{\nu-1}} \sum_{p=0}^{\infty} \frac{(-1)^p\, \omega_p}{p!\, 2^p} \left(\frac{\sqrt{c^2t^2 - r^2}}{\mu}\right)^{p-\nu+1} \\ &\quad \cdot N_{p-\nu+1}\left(\mu\sqrt{c^2t^2 - r^2}\right) \end{aligned} \tag{36}$$

for all values of $c^2t^2 - r^2$.

Taking into account that the *Neumann* function satisfies the *Bessel*'s equation as well as the recurrence relations of the *Neumann* functions one can immediately check that the expression (26) satisfies the equation:

$$\mathcal{H}\mathcal{G}\varphi = 0$$

when the functions ω_p satisfy the recurrence relations (18).

Furthermore $\mathcal{G}\varphi$ reduces to $\Sigma_{2\nu}^{-} - \Sigma_{2\nu}^{+}$ when the electromagnetic field vanishes. It represents therefore the matrix density corresponding to vacuum in the $S_{2\nu}$ world. For $\nu = 2$ one obtains the density of the unobservable distribution in the real four-dimensional world.

We shall return, in a forthcoming publication, on the applications of the expression (26).

Bucharest, September 1940

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Contributions to the theory of positrons (II)

Şerban Ţiţeica

Abstract

The author has obtained in a previous paper the elementary solution of *Dirac*'s wave equation. In the present paper, he uses this result in order to obtain the solution of *Cauchy*'s problem of this equation and to establish some four-dimensional commutation relations.

1

In a previous paper [1], we have shown that the *Hadamard* approach for integration of equations with partial derivatives can be applied to *Dirac* system of equation, describing the motion of a magnetic electron in some electromagnetic field. The goal of the aforementioned paper was to obtain the 'elementary solution' (in *Hadamard*'s sense) for the *Dirac* system, which represents in positron theory the unobservable electron density matrix. When this solution is known, the solution of the *Cauchy* problem for arbitrary initial values given to the wave function, can be obtained with the methods described by *Hadamard*.

The goal of this note is to transpose or, better to directly construct the solution of the *Cauchy* problem by using not the *Hadamard*'s approach, which is not of current use in physics, but the δ-function approach of *Dirac*. The calculation we shall do here will provide us the four-dimentional commutation relations for electron wave functions similar to those obtained by *Jordan* and *Pauli* [2] in their research on the quantization of an electromagnetic field propagating in vacuum.

2

The quantity we are now interested in is the density

$$\Sigma^- + \Sigma^+ = \sum \psi\left(\vec{x'}, t'\right) \psi^*\left(\vec{x''}, t''\right) \tag{1}$$

where we keep the same notations as in [1]; since the sum is taken over all states, irrespective of the energy sign, this density becomes for $t' = t''$ just the *Dirac* δ function. Therefore this density allows us to solve the *Cauchy* problem, as if it is multiplied by a certain function f depending on $\vec{x''}, t''$ (and on spin variable σ'' not explicitly written here) and integrated over $\vec{x''}$ (and summed up over σ''), one obtains a function of $\vec{x'}, t'$ (and σ'), which is a solution of the *Dirac* equation reducing to f for $t' = t''$.

Dirac [3] calculates the sum (1) for a four-dimensional spacetime in absence of the electromagnetic field. In order to apply the method of descent in presence of fields, one has to calculate first this sum for a five-dimensional spacetime. In this case, instead of formulas (11) and (12) of [1] we have the following ones:

$$\Sigma^-_{2\nu} + \Sigma^+_{2\nu} = i\mathcal{G}_0 \frac{1}{(2\pi)^{2\nu-1}} \int e^{i\vec{k}\vec{x}} \frac{\sin\left(ct\sqrt{k^2+\mu^2}\right)}{\sqrt{k^2+\mu^2}} \mathrm{d}\vec{k} \tag{2}$$

and

$$\Sigma^-_{2\nu+1} + \Sigma^+_{2\nu+1} = i\mathcal{N}_0 \frac{1}{(2\pi)^{2\nu}} \int e^{i\left(\vec{k}\vec{x}+lz\right)} \frac{\sin\left(ct\sqrt{k^2+l^2}\right)}{\sqrt{k^2+l^2}} \mathrm{d}\vec{k}\mathrm{d}l \tag{3}$$

We shall first evaluate the integral:

$$I_1' = \frac{1}{(2\pi)^{2\nu}} \int e^{i\left(\vec{k}\vec{x}+lz\right)} \frac{\sin\left(ct\sqrt{k^2+l^2}\right)}{\sqrt{k^2+l^2}} \mathrm{d}\vec{k}\mathrm{d}l.$$

Putting $K = \sqrt{k^2+l^2}$ and $R = \sqrt{x_1^2 + x_2^2 + \cdots + x_{2\nu-1}^2 + z^2}$, and taking the integral over all the directions in the space of coordinates $\vec{k}, l$ one gets (see [1])

$$\begin{aligned} I_1' &= \frac{1}{(2\pi)^\nu} \left(-\frac{1}{R}\frac{\partial}{\partial R}\right)^{\nu-1} \int_0^\infty J_0(KR)\sin(ctK)\,\mathrm{d}K \\ &= \frac{1}{(2\pi)^\nu} \left(-\frac{1}{R}\frac{\partial}{\partial R}\right)^{\nu-1} \frac{\varepsilon}{\sqrt{c^2t^2-R^2}} \end{aligned} \tag{4}$$

with

$$\varepsilon = \begin{cases} 1 & \text{for } ct > 1 \\ 0 & \text{for } -R < ct < R \\ -1 & \text{for } ct < -R \end{cases} \tag{5}$$

The integral

$$I_1 = \frac{1}{(2\pi)^{2\nu-1}} \int e^{i\vec{k}\vec{x}} \frac{\sin\left(ct\sqrt{k^2+\mu^2}\right)}{\sqrt{k^2+\mu^2}} \mathrm{d}\vec{k}$$

can be evaluated by applying to I_1 the descent trick

$$
\begin{aligned}
I_1 &= \int_{-\infty}^{+\infty} I_1' e^{-i\mu z} \mathrm{d}z \\
&= \frac{1}{(2\pi)^{\nu}} \left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^{\nu-1} \varepsilon' \int_{-\sqrt{c^2t^2-r^2}}^{+\sqrt{c^2t^2-r^2}} \frac{e^{-i\mu z}}{\sqrt{c^2t^2-r^2-z^2}} \mathrm{d}z \\
&= \frac{1}{2}\frac{1}{(2\pi)^{\nu-1}} \left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^{\nu-1} \varepsilon' J_0\left(\mu\sqrt{c^2t^2-r^2}\right) \qquad (6)
\end{aligned}
$$

where J_0 is the *Bessel* function and ε' is $+1$ for $ct > r$, -1 for $ct < -r$ and zero for $r > ct > -r$.

For a four-dimensional universe ($\nu = 2$), the formulas (6) take the form given by *Dirac* [3].

3

Before considering the problem in the presence of a field, let us compare the integrals I_1' and I' given by the formulas (4)in the present paper and equation (13) in [1]. The singularity is similar in both cases, with the difference that I_1' is non-zero inside the light cone, while I' is non-zero only outside it.

In passing the case when fields are present we have to replace the solution given by eq. (32) of [1] by an expression with identical singularities, but non-zero inside the cone. A calculation similar to that done in [1] yields for this solution the expression

$$
\omega^{(1)} = \varepsilon \frac{1}{2\pi^{\nu-\frac{1}{2}}} \sum_{p=0}^{\infty} \frac{(-1)^p\,\omega_p}{p!\,2^{2p}\Gamma\left(p-\nu+\frac{3}{2}\right)} \left(c^2t^2-R^2\right)^{p-\nu+\frac{1}{2}}. \qquad (7)
$$

The sum $\Sigma^- + \Sigma^+$ in presence of an external field becomes

$$
\varkappa_1 = i\mathcal{N}\omega^{(1)}. \qquad (8)
$$

For performing the descent one has to calculate first the integrals

$$
B_p = \frac{1}{2^{p-\nu+1}\sqrt{\pi}\Gamma\left(p-\nu+\frac{3}{2}\right)} \int_{-\infty}^{+\infty} \varepsilon \left(c^2t^2-R^2\right)^{p-\nu+\frac{1}{2}} e^{-i\mu z} \mathrm{d}z
$$

$$
= \frac{\varepsilon'}{2^{p-\nu+1}\sqrt{\pi}\Gamma\left(p-\nu+\frac{3}{2}\right)} \int_{-\sqrt{c^2t^2-r^2}}^{+\sqrt{c^2t^2-r^2}} \left(c^2t^2-r^2-z^2\right)^{p-\nu+\frac{1}{2}} e^{-i\mu z} \mathrm{d}z.
$$

If $p \geq \nu - 1$ these integrals are convergent and can be expressed in terms of *Bessel* functions

$$
B_p = \varepsilon' \left(\frac{\sqrt{c^2t^2-r^2}}{\mu}\right)^{p-\nu+1} J_{p-\nu+1}\left(\mu\sqrt{c^2t^2-r^2}\right). \qquad (9)
$$

They obey the recurrence relation

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)B_{p+1}=B_p, \tag{10}$$

which is obtained by taking into account the recurrence relations of *Bessel* functions and by observing that the derivative of ε' is a δ function on the light cone where the *Bessel* function vanishes. Such a product is identically zero and one can treat ε' as a constant.

For $p<\nu-1$, we shall define [4] the integrals B_p by the formula (10)

$$\begin{aligned} B_{\nu-2} &= \left(-\frac{1}{r}\frac{\partial}{\partial r}\right)B_{\nu-1}=\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)\varepsilon' J_0\left(\mu\sqrt{c^2t^2-r^2}\right) \\ &= \varepsilon'\left(\frac{\sqrt{c^2t^2-r^2}}{\mu}\right)^{-1}J_{-1}\left(\mu\sqrt{c^2t^2-r^2}\right)+J_0\left(-\frac{1}{r}\frac{\partial\varepsilon'}{\partial r}\right). \end{aligned} \tag{11}$$

However

$$\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)\varepsilon'=\frac{\delta\left(ct-r\right)-\delta\left(ct+r\right)}{r}=\Delta,$$

where Δ is the singular function introduced by *Jordan* and *Pauli* [2]; moreover, the factor multiplying this singular function is different from zero on the light cone, and so does the last term of (11). Nevertheless it is possible to replace J_0 by its value on the light cone, i.e by one. So, we have

$$B_{\nu-2}=\varepsilon'\left(\frac{\sqrt{c^2t^2-r^2}}{\mu}\right)^{-1}J_{-1}\left(\mu\sqrt{c^2t^2-r^2}\right)+\Delta. \tag{12}$$

Similarly, one obtains

$$\begin{aligned} B_{\nu-1-\lambda} &= \varepsilon'\left(\frac{\sqrt{c^2t^2-r^2}}{\mu}\right)^{-\lambda}J_{-\lambda}\left(\mu\sqrt{c^2t^2-r^2}\right) \\ &\quad+\sum_{p=0}^{\lambda-1}(-1)^p\,\frac{\mu^{2p}}{p!\,2^p}\left(-\frac{1}{r}\frac{\partial}{\partial r}\right)^{\lambda-1-p}\Delta, \end{aligned} \tag{13}$$

for $\lambda=2,3,\ldots,\nu-1$.

With these values we are able to calculate the integral

$$\varphi_1=\int_{-\infty}^{+\infty}\omega^{(1)}e^{-i\mu z}\mathrm{d}z. \tag{14}$$

For this purpose we replace in $\omega^{(1)}$ in (14) by its expansion (5) and integrate the result term by term. One gets

$$\varphi_1=\frac{1}{2\left(2\pi\right)^{\nu-1}}\sum_{p=0}^{\infty}\frac{(-1)^p\,\omega_p}{p!\,2^p}B_p,$$

which becomes

$$\varphi_1 = \frac{1}{2\,(2\pi)^{\nu-1}} \left[\varepsilon' \sum_{p=0}^{\infty} \frac{(-1)^p\,\omega_p}{p!\,2^p} \left(\frac{\sqrt{c^2t^2 - r^2}}{\mu} \right)^{p-\nu+1} J_{p-\nu+1}\left(\mu\sqrt{c^2t^2-r^2}\right) \right.$$

$$\left. + \sum_{p=0}^{\nu-2} \frac{(-1)^p\,\omega_p}{p!\,2^p} \sum_{q=0}^{\nu-2-p} (-1)^q\, \frac{\mu^{2q}}{q!\,2^q} \left(-\frac{1}{r}\frac{\partial}{\partial r} \right)^{\nu-2-p-q} \Delta \right] \tag{15}$$

with the expressions previously found for the integrals B_p.

The function

$$\psi_1 = i\mathcal{G}\varphi_1 \tag{16}$$

with φ_1 given by (15), is the matrix corresponding to $\Sigma^- + \Sigma^+$ when the field is present: considered as a function of $\vec{x'}$ and t' it is a solution of the *Dirac* equation; moreover for $t' = t''$ it reduces to the $\delta\left(\vec{x}\right) = \delta\left(\vec{x'} - \vec{x''}\right)$-function (multiplied by a δ function with respect to the spin variables). This is the function staying on the right hand side of the commutation relations for the *Dirac* functions taken at two arbitrary points of the spacetime

$$\psi\left(\vec{x'}, t'\right)\psi^*\left(\vec{x''}, t''\right) + \psi^*\left(\vec{x''}, t''\right)\psi\left(\vec{x'}, t'\right) = \psi\left(\vec{x'}, t'; \vec{x''}, t''\right).$$

This is also the function to be used in order to solve the Cauchy problem for Dirac equation.

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On the proper time in wave mechanics

Serban Titeica

The proper time τ of a material point is defined in relativistic classical (non-quantum) mechanics by the relation:

$$d\tau = \sqrt{dt^2 - \frac{1}{c^2}\sum_1^3 dx_i^2} \tag{1}$$

where $dt,\ dx_1,\ dx_2,\ dx_3$ are the components of a world line described by the respective point and c is the speed of light. If the material point has an electric charge and is subjected only to the action of an electromagnetic field given by the scalar potential V and the vector potential $A_1,\ A_2,\ A_3$, the world lines are those rendering the integral

$$\int L dt = \int \left[-mc^2 \sqrt{1 - \frac{1}{c^2}\sum \left(\frac{dx_i}{dt}\right)^2} + \frac{\mathrm{e}}{c}\sum A_i \frac{dx_i}{dt} - \mathrm{e}V \right] dt \tag{2}$$

stationary; m is the rest mass, and e, the charge of the material point. To this problem of variational calculus corresponds a system of canonical equations having as hamiltonian function, the function:

$$H = \mathrm{e}V + c\sqrt{m^2c^2 + \sum \left(p_i - \frac{\mathrm{e}}{c}A_i\right)^2} \tag{3}$$

The definition (1) cannot be used in wave mechanics, as the concept of world line makes here no sense. The transition from classical to wave mechanics can be done by the usual procedures only if we can create a canonical system where the proper time is explicitly present. It is possible to choose τ as *independent variable*; then, the canonical system will correspond to the following variational problem:

$$\delta \int \mathcal{L} d\tau = \delta \int L \cdot \frac{dt}{d\tau} \cdot d\tau = 0. \tag{4}$$

We shall not insist on this procedure of introducing the proper time in wave mechanics, as it was already discussed in several papers [1]. We shall only stress that one must always take into consideration an additional condition, equivalent to the relation

$$\left(\frac{dt}{d\tau}\right)^2 - \frac{1}{c^2}\sum_1^3 \left(\frac{dx_i}{d\tau}\right)^2 = 1 \tag{5}$$

expressed in terms of canonical variables.

There is another approach for obtaining a canonical system containing τ, which apparently has not been yet noticed till now, and which is much more convenient for practical applications: one maintains t as an independent variable and one considers τ as a functions of t. Let us add to the canonical system associated with hamiltonian function (3)

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial p_i},\ \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i} \tag{6}$$

the eq. (1)

$$\frac{d\tau}{dt} = \sqrt{1 - \frac{1}{c^2}\sum \left(\frac{dx_i}{dt}\right)^2} = \frac{mc}{\sqrt{m^2c^2 + \sum \left(p_i - \frac{\mathrm{e}}{c}A_i\right)^2}}$$

where the quantities $\frac{dx_i}{dt}$ of the second term were replaced by their values given by (6). From a formal point of view, this equation could be written as

$$\frac{d\tau}{dt} = \frac{\partial H}{\partial \left(mc^2\right)},$$

and completed by the conjugate equation

$$\frac{d\left(mc^2\right)}{dt} = -\frac{\partial H}{\partial \tau} = 0,$$

as H does not depend on τ. This formal remark allows us to construct a canonical system containing the proper time as follows: we start with the hamiltonian function

$$\mathcal{H} = \mathrm{e}V + \sqrt{P^2 + c^2 \sum \left(p_i - \frac{\mathrm{e}}{c} A_i\right)^2}$$

which can be obtained from (3) replacing mc^2 by a new variable P canonically conjugate to a new coordinate τ which does not enter explicitly in $\mathcal{H}$, and write down the associated canonical system. This system admits the integral $P = const$ and reduces to the canonical system corresponding to the function H and to eq.(1) defining the proper time, if the supplementary condition

$$P = mc^2 \tag{7}$$

is satisfied.

Let us pass now to wave mechanics. The previous recipe for passing from H to $\mathcal{H}$ can be immediately transferred in quantum terms: we simply must replace in the quantum expression of H the product mc^2 by the quantum operator corresponding to P, namely $\frac{h}{2\pi i}\frac{\partial}{\partial \tau}$. Assuming, for simplicity, that the electromagnetic field is zero, Dirac equation becomes, after this substitution

$$\frac{1}{c}\frac{\partial \psi}{\partial t} + \vec{\alpha} \bigtriangledown \psi + \beta \frac{1}{c}\frac{\partial \psi}{\partial \tau} = 0.$$

Let us notice, by the way, that it is more convenient to introduce the world line length $s = c\tau$ instead of τ. Actually, it was this equation, completed with terms containing the electrodynamic potentials, the one we used [2] in our research on positron theory. The method of descent used by us can be viewed in quantum terms as a transition from the representation where τ is diagonal to that where the conjugate variable is diagonal and has the values imposed by eq. (7).

For the applications of the substitution $mc^2 \to \frac{h}{2\pi i}\frac{\partial}{\partial \tau}$ to the wave equation of particles with spin different from 1/2 one might refer to Hund's paper [3].

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The vacuum polarization

Şerban Ţiţeica

Abstract

On the basis of an earlier proposal [7] concerning subtraction terms in the hole theory the vacuum polarization in the lowest order of a fine structure constant expansion is calculated. It turns out that the positron theory does not lead to linear modifications of the *Maxwell* equations.

1

. The polarization of a dielectric is due, as we know, to the action of an external field on the electric charges already existent in the dielectric, but they are usually distributed in such a way, that their existence is not directly observable. *Dirac*'s theory of 'holes' makes possible to predict that the vacuum might also allow for a similar polarization; indeed, by a mechanism similar to that of pair creation of particles, an electromagnetic field could modify the distribution of electrons over the negative energy levels in such a way, that observable charges could show up. Similar to the polarization of dielectrics, and for the same reasons, this phenomenon has as consequence a modification of *Maxwell* equations. From a theoretical point of view, the importance of this phenomenon (whose existence has been not yet confirmed by experiment)consists in the fact that it could provide some information about the subtractive terms in the theory of positrons; on this subject, several studies have been already published, see [3] and [4]. The aim of this article is to calculate the charges induced in vacuum by an electromagnetic field using the methods developed in a previous work [7]. The first two sections provide a summary of the results of this just cited work; the third one attempts to solve the particular problem we are interested in.

2

In wave mechanics, the right method for studying the systems composed of an arbitrary number of particles is the superquantization. For the problem of vacuum polarization, it suffices to limit ourselves to the *Hartree* approximation. Taking into account the fact that the electrons obey the exclusion principle, the properties of the system can be obtained by considering the sum

$$\sum_{\rm oc} \psi_{\rm n}\left(x', t\right) \psi^*_{\rm n}\left(x'', t\right) \tag{1}$$

where 'oc' means that the sum is extended over all the states n occupied by electrons; this sum is the quantum mechanical equivalent of the distribution function used in classical statistical mechanics. In a relativistic wave theory of the electron, we have to take into account that the wave function has four components $\psi(x, t, k)$ with $k = 1, 2, 3, 4$, and that it is necessary to introduce two distinct values, t' and t'' for the time, as if t' and t'' were be the same, the equality of both values would destroy the symmetry between space and time; we are thus lead to consider the sum

$$\sum_{\rm oc} \psi_{\rm n}\left(x', t', k'\right) \psi^*_{\rm n}\left(x'', t'', k''\right). \tag{2}$$

According to the hypothesis of holes, a distribution with all negative energy states occupied, and all positive energy states unoccupied is unobservable; in order to obtain the sum which corresponds to observable quantities, we have to subtract from Eq.(2) the sum

$$\sum_{(-)} \psi_{\rm n}\left(x', t', k'\right) \psi^*_{\rm n}\left(x'', t'', k''\right);$$

with the result

$$r = \sum_{(+)\rm oc} - \sum_{(-)\rm unoc}, \tag{3}$$

and all average values of observable quantities have to be calculated using r.

Let us introduce now

$$\begin{aligned} R_s &= \frac{1}{2}\left(\sum_{\rm oc} - \sum_{\rm unoc}\right), \\ S &= \frac{1}{2}\left(\sum_{(-)} - \sum_{(+)}\right); \end{aligned} \tag{4}$$

so we can write:

$$r = R_s - S. \tag{5}$$

Accordingly, the calculation of r can be done in two stages: we first calculate R_s that creates, in general, only technical difficulties, as for any correctly formulated physical problem, we know which states are occupied and which are not. On the contrary, the calculation of the subtractive term S presents difficulties of principle since in the presence of an electromagnetic field, the distinction between the positive energy levels and the negative energy ones becomes illusory.

In order to find S, the following remark, due to *Dirac*, is crucial [1]: the sum R_s has, for values of x', t', x'', t'' related by $\sum\left(x'_i - x''_i\right)^2 - c^2\left(t' - t''\right)^2 = 0$, characteristic algebraic-logarithmic singularities. S must have, therefore, the same singularities since the difference r is regular. This leads *Dirac* to choose as value of S the sum of singular terms in R_s; this sum is perfectly determined for a given electromagnetic field. *Heisenberg* [2] slightly modifies *Dirac*'s choice by adding to the singular terms a regular one, chosen in such a way, that the total electric charge induced by vacuum polarization be zero.

Let us notice now that the values of S given by *Heisenberg* do not satisfy the *Dirac*'s wave equation, and so does not r. *Heisenberg* himself stresses that this fact is the mathematical expression for the possibility of creating and annihilating particles. However, the expression Eq.(4) shows that S has to satisfy the *Dirac* equation with respect to the variables x', t', k' and the adjoint equation with respect to the variables x'', t'', k''. Consequently, it seems to us that a second condition should be imposed to S, namely to satisfy the wave equation; notice that this does not contradict the possibility of pair creation as shown by the usual theory of *Klein* paradox.

These two conditions satisfied by S do not yet completely determine this function. From the physical point of view this is normal; as, if the situation were opposite the function R_s satisfying the same conditions as S would be equal to S, so r would be identically zero. From a mathematical point of view, this means just generalizing to the *Dirac* equation the result obtained by *Hadamard* [8] for the 'fundamental' solutions of hyperbolic second order equations: when the number of independent variables is even, the fundamental solution is not determined by its singularities.

So we have to choose among the solutions of *Dirac* equation having the appropriate singularity, the one to be identified with S. In our paper already cited, we proposed the following choice, inspired by the works of *Hadamard*: one introduces in the *Dirac* equation an additional independent variable s, replacing the term mc by $\frac{\hbar}{i}\frac{\partial}{\partial s}$, where $\hbar$ is the *Planck constant divided by* 2π; one looks for a fundamental solution of this new equation with an odd number of independent variables; this solution is completely determined by its singularities, as has been demonstrated by *Hadamard* for second order equations; one comes then back to the real, four-dimensional spacetime, by applying the 'descent' trick; the result is *by assumption* equal to S.

It seems difficult to find a physical interpretation of this purely formal procedure. Until now we succeeded only in giving the physical interpretation for the variable s and for the method of descent: s is the proper time multiplied by the speed of light, and mc is its conjugated momentum [10]. The descent is the transition to a representation of wave function in which the momentum is diagonal and has exactly the value mc.

3

Let us briefly expose now the mathematical calculations, our goal being mainly to introduce a more symmetrical and more convenient notation. For details, the reader may consult our paper already cited.

The *Dirac* equation with an additional variable reads:

$$\frac{1}{c}\frac{\partial\psi}{\partial t} - \frac{ie}{\hbar c}A_0 + \vec{\alpha}\left(\nabla + \frac{ie}{\hbar c}\vec{A}\right)\psi + \beta\frac{\partial\psi}{\partial s} = 0 \tag{6}$$

Multiplying from left by β and utilizing the usual notations [9], one finds

$$\frac{\partial\psi}{\partial s} + i\mathcal{H}\psi = 0 \tag{7}$$

with

$$\mathcal{H} = \left(\gamma\frac{\partial}{\partial x}\right) + \frac{ie}{\hbar c}\left(\gamma\Phi\right) = \sum_1^4 \gamma_\rho\frac{\partial}{\partial x_\rho} + \frac{ie}{\hbar c}\sum_1^4 \gamma_\rho\Phi_\rho. \tag{8}$$

Let us notice that the adjoint of Eq.(7) is not any more satisfied by the complex conjugate of ψ, but by the function $\psi^+ = i\psi^*\gamma_4$. The fundamental solution of Eq.(7) does not correspond any more to a sum of terms having the form $\psi\left(x', s', k'\right)\psi^*\left(x'', s'', k''\right)$, but to one of the form $\psi\left(x', s', k'\right)\psi^+\left(x'', s'', k''\right)$; it plays exactly the role of the previous one, and one can switch from one to another through a simple multiplication by γ_4.

We now look for a function $\psi\left(x', s', k';\ x'', s'', k''\right)$ which, taken as a function of x', s', k' satisfies Eq.(7) and, on the light cone

$$D \equiv \left(s' - s''\right)^2 + \sum_1^4 \left(x'_\rho - x''_\rho\right)^2 = 0 \tag{9}$$

tends to infinity as $D^{-\frac{5}{2}}$.

Let us write

$$\psi = \left(s' - s''\right)U + V \tag{10}$$

where U and V are functions of $D,\ x',\ k',\ x'',\ k''$. Substituting this expression into Eq.(7) and eliminating everywhere the even powers of $(s' - s'')$ by using D, one finds

$$\begin{aligned} U + 2\left(D - R^2\right)\frac{\partial U}{\partial D} + 2\left(s' - s''\right)\frac{\partial V}{\partial D} & \\ + 2i\left(\gamma x\right)\left[\left(s' - s''\right)\frac{\partial U}{\partial D} + \frac{\partial V}{\partial D}\right] & \\ + i\mathcal{H}'\left[\left(s' - s''\right)U + V\right] = 0 & \end{aligned}$$

where $x_\rho = x'_\rho - x''_\rho$, $s = s' - s''$, $R^2 = x_1^2 + \cdots + x_4^2$ and the accent on $\mathcal{H}'$ reminds that this operator acts on the variables x', k'; moreover, the derivatives with respect to coordinates x' have to be calculated by treating D as a constant. Separating the first order terms in s from those which do not contain this factor, we obtain the following system of equations:

$$\begin{aligned} U + 2\left(D - R^2\right)\frac{\partial U}{\partial D} + 2i\left(\gamma x\right)\frac{\partial V}{\partial D} + i\mathcal{H}'V &= 0 \\ 2\frac{\partial V}{\partial D} + 2i\left(\gamma x\right)\frac{\partial U}{\partial D} + i\mathcal{H}'U &= 0. \end{aligned} \tag{11}$$

one can eliminate V among these equations in order to obtain an equation with only one unknown function, U

$$2D\frac{\partial^2 U}{\partial D^2}+\left[7+2\left(x\frac{\partial}{\partial x'}\right)+2\frac{i\mathrm{e}}{\hbar c}\left(x\Phi'\right)\right]\frac{\partial U}{\partial D}+\frac{1}{2}\mathcal{H}'^2 U=0. \tag{12}$$

We now write U in a form showing explicitly its singularity

$$U=uD^{-\frac{5}{2}}, \tag{13}$$

where u is a regular function. By substituting Eq.(13) into Eq.(12), we get

$$\begin{aligned}2\frac{\partial^2 u}{\partial D^2}+\frac{1}{D}\left[-3+2\left(x\frac{\partial}{\partial x'}\right)+2\frac{i\mathrm{e}}{\hbar c}\left(x\Phi'\right)\right]\frac{\partial u}{\partial D}\\ -\frac{5}{D^2}\left[\left(x\frac{\partial}{\partial x'}\right)+\frac{i\mathrm{e}}{\hbar c}\left(x\Phi'\right)\right]u+\frac{1}{2D}\mathcal{H}'^2 U=0\ .\end{aligned}$$

Let us look for a solution of this equation having the form

$$u=u_0+u_1 D+\cdots+u_n D^n+\cdots. \tag{14}$$

For $n=1,2,3,\ldots$ we find the recurrence relations

$$\left[\left(x\frac{\partial}{\partial x'}\right)+\frac{i\mathrm{e}}{\hbar c}\left(x\Phi'\right)+n\right]u_n=-\frac{1}{2\left(2n-5\right)}\mathcal{H}'^2 u_{n-1} \tag{15}$$

and for $n=0$ the equation

$$\left[\left(x\frac{\partial}{\partial x'}\right)+\frac{i\mathrm{e}}{\hbar c}\left(x\Phi'\right)\right]u_0=0. \tag{16}$$

This last equation and the regularity condition determine u_0 up to a constant factor which, according to the calculations done in our paper, previously cited, has to be chosen such that for a zero field ψ reduces to the sum

$$\frac{1}{2}\left(\sum_{(-)}-\sum_{(+)}\right).$$

One finds

$$u_0=\frac{3}{8\pi^2}\exp\left[-\frac{i\mathrm{e}}{\hbar c}\int_{x''}^{x'}\Phi\mathrm{d}x\right] \tag{17}$$

where the integration path is the line segment joining the spacetime points x' and x''.

In order to obtain V as a solution of the second Eq.(11) we write

$$V=vD^{-\frac{5}{2}}=D^{-\frac{5}{2}}\sum_{0}^{\infty}v_n D^n, \tag{18}$$

and get the following relations:

$$\begin{aligned} v_0 &= -i\,(\gamma x)\,u_0 \\ v_n &= -i\,(\gamma x)\,u_n - \frac{i}{2n-5}\mathcal{H}' u_{n-1}, \end{aligned} \tag{19}$$

for $n = 1, 2, 3, ...$

The recurrence relations Eqs.(15) and (19) are easier to handle if we put

$$\begin{aligned} u_n &= u_0 \bar{u}_n \\ v_n &= u_0 \bar{v}_n, \end{aligned} \tag{20}$$

in which case, they become

$$\begin{aligned} \left[x\frac{\partial}{\partial x'} + n\right]\bar{u}_n &= -\frac{1}{2\,(2n-5)}\mathcal{G}'^2 \bar{u}_{n-1}, \\ \bar{v}_n &= -i\,(\gamma x)\,\bar{u}_n - \frac{i}{2n-5}\mathcal{G}' \bar{u}_{n-1} \end{aligned} \tag{21}$$

where $\mathcal{G}'$ is the result obtained by replacing in $\mathcal{H}'$ the potentials Φ' by $\Phi' - \dfrac{\partial \lambda}{\partial x'}$ with $\lambda = \displaystyle\int_{x''}^{x'} (\Phi \mathrm{d}x)$.

Let us remind now that in the method of descent one multiplies both sides of Eq.(10) by $\exp(-i\mu s)$, where $\mu = \dfrac{mc}{\hbar}$, and one integrates over s along a contour conveniently chosen in the complex plane of this variable . We have demonstrated [7] that the result can be expressed in terms of *Neumann* functions, singular solutions of the *Bessel*'s equation. We shall give here only the most important terms of the expansion of these functions, limiting ourselves to the case $x_4 = 0$

$$\begin{aligned} \int D^{-\frac{5}{2}} \exp(-i\mu s)\,\mathrm{d}s &= -\frac{\mu^4}{12}\left[\frac{1}{2}\ln\left(\frac{\gamma\mu r}{2}\right)^2 + \left(\frac{2}{\mu r}\right)^2 - \left(\frac{2}{\mu r}\right)^4 - \frac{3}{4}\right] \\ &\quad + o\,(r) \\ \int D^{-\frac{3}{2}} \exp(-i\mu s)\,\mathrm{d}s &= \frac{\mu^2}{2}\left[\frac{1}{2}\ln\left(\frac{\gamma\mu r}{2}\right)^2 + \left(\frac{2}{\mu r}\right)^2 - 1\right] + o\,(r) \\ \int D^{-\frac{1}{2}} \exp(-i\mu s)\,\mathrm{d}s &= -\ln\left(\frac{\gamma\mu r}{2}\right)^2 + o\,(r) \\ \int D^{n-\frac{1}{2}} \exp(-i\mu s)\,\mathrm{d}s &= (-1)^n\,\frac{2}{\mu^{2n}}\,(2n-1)! + o\,(r) \end{aligned} \tag{22}$$

for $n = 1, 2, \ldots$ and where $r^2 = x_1^2 + x_2^2 + x_3^2$, $\gamma = e^C$, C being the *Euler*'s constant $C = 0.577\ldots$ (one should not confuse γ with the 4×4 matrices); $o\,(r)$ designs terms vanishing as $r \to 0$.

4

Let us now explicitly compute the fundamental solution up to the first order in e$/\hbar c$, that is linear in the potentials Φ. We take advantage of a remark by *Pauli* and *Rose* [5], and limit ourselves, without restricting the generality, to the case when Φ functions are plane waves. We set

$$\Phi_\rho = \varphi_\rho \exp\left[i\left(p_1 x_1 + ... + p_4 x_4\right)\right] = \varphi_\rho e^{i\,(px)} \tag{23}$$

with $\varphi = const$, and obtain

$$\lambda = \int_{x''}^{x'} (\Phi \mathrm{d}x) = (x\varphi)\,\frac{\sin X}{X} e^{i\,(p\xi)} \tag{24}$$

where the following notations: $x_\rho = x'_\rho - x''_\rho$, $\xi_\rho = \dfrac{x'_\rho + x''_\rho}{2}$, and $X = \left(\dfrac{px}{2}\right)$ were introduced. The recurrence relations Eq.(21) for $\bar{u}_n$ become after neglecting the powers of e$/\hbar c$ higher than the first

$$\begin{aligned}
\left(x\frac{\partial}{\partial x'} + n\right)\bar{u}_n &= -\frac{1}{2\,(2n-5)}\nabla'^2 \bar{u}_{n-1} \quad \text{for } n = 2,\ 3, \ldots \\
\left(x\frac{\partial}{\partial x'} + 1\right)\bar{u}_1 &= \frac{i\mathrm{e}}{6\hbar c} e^{i\,(p\xi)} \left\{\left[X\,(p\varphi) - \frac{p^2}{2}\,(x\varphi)\right]\left(i\frac{e^{iX}}{X} - \frac{e^{iX}}{X^2}\right.\right. \\
&\qquad \left.\left. + \frac{\sin X}{X^3}\right) + i e^{iX} \sum_{\rho>\sigma} \gamma_\rho\gamma_\sigma\left(p_\rho\varphi_\sigma - p_\sigma\varphi_\rho\right)\right\}.
\end{aligned}$$

It is easy to check by using the recurrence relations for the *Bessel* functions $J_\nu\,(X)$ that the above differential equations have as solutions the expressions

$$\begin{aligned}
\bar{u}_n &= \frac{i\pi\mathrm{e}}{12\hbar c}\frac{e^{i\,(p\xi)}}{\Gamma\left(n-\frac{3}{2}\right)}\left(\frac{p^2}{16}\right)^{n-1}\left\{\left[X\,(p\varphi) - \frac{p^2}{2}\,(x\varphi)\right]F_n\,(X)\right. \\
&\qquad \left. -2iF_{n-1}\,(X)\sum_{\rho>\sigma}\gamma_\rho\gamma_\sigma\left(p_\rho\varphi_\sigma - p_\sigma\varphi_\rho\right)\right\}
\end{aligned} \tag{25}$$

for $n = 1, 2, \ldots$, and where

$$F_n\,(X) = \frac{J_{n+\frac{1}{2}}\,(X)}{\left(\dfrac{X}{2}\right)^{n+\frac{1}{2}}}$$

Let us consider now the second relation Eq.(21) in order to calculate the coefficients $\bar{v}_n$. One can immediately see that these functions contain terms of first and third degree with respect to γ matrices. The last ones are not of interest for our problem, because, in order to write down the matrix density of the charge and current, one multiplies by γ_ρ and one takes the trace with respect to spin variables; or, for all third degree terms, the result of this operation is zero. Keeping only the first order terms in γ we get

$$
\begin{aligned}
\bar{v}_0 &= -i\,(\gamma x)\\
\bar{v}_n &= \frac{\pi \mathrm{e}}{24\hbar c}\frac{e^{i\,(p\xi)}}{\Gamma\left(n-\frac{3}{2}\right)}\left(\frac{p^2}{16}\right)^{n-2}\left\{\frac{p^2}{4}\left[\frac{1}{2}(\gamma p)(x\varphi)X+\frac{1}{2}(\gamma x)(p\varphi)X\right.\right.\\
&\left.-\frac{1}{4}p^2(\gamma x)(x\varphi)-X^2(\gamma\varphi)\right]F_n(X)\\
&\left.+(n-1)\left[p^2(\gamma\varphi)-(\gamma p)(p\varphi)\right]F_{n-1}(X)\right\} \qquad (26)
\end{aligned}
$$

with $n = 1, 2, ...$

One more simplification occurs if we consider the case when only the scalar potential is non zero, and we calculate the matrix of the charge density only; through a *Lorentz* transformation we can go from this particular case, to the general one. So, we have to multiply the fundamental solution by γ_4 and take the trace with respect to spin variables. Only the terms of first order in γ_4 gives a result different of zero (and equal to 4). Moreover, let us take $t' = t''$, i.e., $x_4 = 0$. We then have

$$
\begin{aligned}
Tr\,(\gamma_4\bar{u}_n) &= 0,\\
Tr\,(\gamma_4\bar{v}_0) &= 0,\\
Tr\,(\gamma_4\bar{v}_n) &= \frac{\pi \mathrm{e}}{24\hbar c}\frac{e^{i\,(p\xi)}}{\Gamma\left(n-\frac{3}{2}\right)}\left(\frac{p^2}{16}\right)^{n-2}\left[-p^2\varphi_4X^2F_n(X)\right.\\
&\left.+4(n-1)\left(p^2-p_4^2\right)\varphi_4F_{n-1}(X)\right] \qquad (27)
\end{aligned}
$$

where X reduces now to $(x_1p_1 + x_2p_2 + x_3p_3)/2$. We pass from $\bar{v}_n$ to v_n by multiplying it by u_0. Or, as $\bar{v}_n$ already contains the factor $\mathrm{e}/\hbar c$, we can, in the expression Eq.(17) of u_0, replace the exponential by one. The subtractive terms of the matrix density of the charge (or, more correctly, of the fourth component, always imaginary, of current four-vector) become, in a spacetime with five dimensions

$$
\begin{aligned}
&-\frac{\mathrm{e}^2}{64\pi\hbar c}\varphi_4e^{i\,(p\xi)}\sum_1^\infty\frac{D^{n-\frac{5}{2}}}{\Gamma\left(n-\frac{3}{2}\right)}\left(\frac{p^2}{16}\right)^{n-2}\left[-p^2X^2F_n(X)\right.\\
&\left.+4(n-1)\left(p^2-p_4^2\right)F_{n-1}(X)\right]. \qquad (28)
\end{aligned}
$$

By descending to the four-dimensional world we neglect the terms vanishing as the vector $\vec{r}$, of components x_1, x_2, x_3 tends to zero. The formulas in Eq.(22) allow us to write

$$
\frac{\mathrm{e}^2}{4\pi^2\hbar c}\varphi_4 e^{i\,(p\xi)}\left[-\frac{1}{3}\frac{(\vec{p}\cdot\vec{x})^2}{r^2}+\frac{1}{3}\,|\vec{p}|^2\ln\left(\frac{\gamma\mu r}{2}\right)^2\right.
$$
$$
\left.+\,2|\vec{p}|^2\sum_0^\infty(-1)^n\,\frac{B\,(n+3,n+3)}{n+1}\left(\frac{p^2}{\mu^2}\right)^{n+1}\right] \tag{29}
$$

where B is the *Euler*ian integral of first kind and $\vec{p}$ the vector with components p_1, p_2, p_3.

The first two terms of the expression Eq.(29) correspond exactly to subtractive terms of the *Heisenberg* theory, Eq.(38) in [2] corrected by Eq.(13) in [6], while the sum Σ corresponds to the regular term of R_s, as one can find it, for instance, from the series expansion of the expression Eq.(21) in the paper by *Pauli* and *Rose* [5]. If one agrees with our assumption concerning the form of subtractive terms, one has to subtract from R_s the whole expression Eq.(29). It results that r is zero. So, if our hypothesis is correct, no linear modification of Maxwell equations follows from the hole theory.

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